11716. *Proposed by Oliver Knill, Harvard University, Cambridge, MA.* Let $\alpha = (\sqrt{5}-1)/2$. Let p_n and q_n be the numerator and denominator of the nth continued fraction convergent to α. (Thus, p_n is the nth Fibonacci number and $q_n = p_{n+1}$). Show that

$$\sqrt{5}\left(\alpha - \frac{p_n}{q_n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{q_n^{2k+2} 5^k},$$

where C_k denotes the kth *Catalan number*, given by $C_k = \frac{(2k)!}{(k!(k+1)!)}$.

11717. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.* Given a circle c and line segment AB tangent to c at a point E that lies strictly between A and B, provide a compass and straightedge construction of the circle through A and B to which c is internally tangent.

11718. *Proposed by Arkady Alt, San Jose, CA.* Given positive real numbers $a_1, \dots, a_n$ with $n \ge 2$, minimize $\sum_{i=1}^n x_i$ subject to the conditions that $x_1, \dots, x_n$ are positive and that $\prod_{i=1}^n x_i = \sum_{i=1}^n a_i x_i$.

SOLUTIONS

A Polygon Equation

11595 [2011, 747]. *Proposed by Victor K. Ohanyan, Yerevan, Armenia.* Let $P_1, \dots, P_n$ be the vertices of a convex n-gon in the plane. Let Q be a point in the interior of the n-gon, and let $\mathbf{v}$ be a vector in the plane. Let $\mathbf{r}_i$ denote the vector QP_i, with length r_i. Let Q_i be the (radian) measure of the angle between $\mathbf{v}$ and $\mathbf{r}_i$, and let F_i and Y_i be, respectively, the clockwise and counterclockwise angles into which the interior angle at P_i of the polygon is divided by QP_i. Show that

$$\sum_{i=1}^{n} \frac{1}{r_i} \sin(Q_i)(\cot F_i + \cot Y_i) = 0.$$

Solution by O. P. Lossers, The Netherlands. We assume without loss of generality that $\mathbf{v}$ is a unit vector. Let $\mathbf{k}$ be a unit vector in three-space orthogonal to the plane of the polygon. Note that $\sin(Q_i)\,\mathbf{k} = \frac{1}{r_i}(\mathbf{r}_i \times \mathbf{v})$. We have

$$\cot F_i = \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times (\mathbf{r}_{i+1} - \mathbf{r}_i)\|} \quad \text{and} \quad \cot Y_i = \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i-1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times (\mathbf{r}_{i-1} - \mathbf{r}_i)\|}$$

(subscripts are taken modulo n). Since $\mathbf{v}$ is arbitrary and $\mathbf{r}_i \times \mathbf{r}_i = \mathbf{0}$, we must prove that

$$\sum_{i=1}^{n} \left(\frac{\mathbf{r}_i \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times \mathbf{r}_{i+1}\|} + \frac{\mathbf{r}_i \cdot (\mathbf{r}_{i-1} - \mathbf{r}_i)}{\|\mathbf{r}_i \times \mathbf{r}_{i-1}\|} \right) \frac{\mathbf{r}_i}{r_i^2} = \mathbf{0}.$$

For geometric reasons, the vector s_i defined by

$$\mathbf{s}_i = \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\|\mathbf{r}_i \times \mathbf{r}_{i+1}\|} + \frac{\mathbf{r}_{i-1} - \mathbf{r}_i}{\|\mathbf{r}_i \times \mathbf{r}_{i-1}\|}$$